Improved Component-Mode Representation for Structural Dynamic Analysis

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The new method described in this paper employs an incomplete set of free-boundary normal modes of vibration, augmented with a low-frequency account for the contribution of neglected (residual) modes. The "residual effects" improve the accuracy of forced dynamic response in a manner which is related to the benefit of the mode-acceleration method. The new method adds residual inertial and dissipative effects to the residual flexibility introduced by MacNeal. All effects are derived from the solution of a special statics problem, followed by removal of the contributions of the retained modes. A structural component can then be represented in a stiffness-matrix form for various applications, one of which is modal synthesis. Numerical results of modal analysis for a cantilevered rod show the new method to yield superior accuracy to several other methods (including those of Hurty, Bamford, and MacNeal). All parameters for the new representation can be derived from test; this is not true for most other methods. Required are the free-boundary normal modes and the dynamic flexibility matrix vs frequency for the boundary points. Consequently, any desired mix of analytically derived and experimentally derived parameters can be employed.

Nomenclature†

A	= matrix transformation, Eq. (17)
B	= damping coefficient in dynamic flexibility, Eq. (23)
c,C	= viscous damping coefficient
E	= modulus of elasticity
f,F	= force
g,G	= flexibility (no rigid-body contributions), Eq. (21)
h,H	= inertial coefficient in dynamic flexibility, Eq. (23)
i	= imaginary unit, $(-1)^{1/2}$
I	= unit matrix
k,K	= stiffness
ℓ	= length
m,M	= mass
$\bar{m}$	= number of discrete coordinates
$\bar{n}$	= number of flexible-body modes
q,Q	= generalized displacement
$\bar{r}$	= number of rigid-body modes
s	= Laplace variable
S	= area
u,U	= displacement
β_ϕ, β_ψ, β_ω	= error exponent for ϵ_ϕ, ϵ_ψ, ϵ_ω; Eq. (44)
ϵ_ϕ, ϵ_ψ, ϵ_ω	= errors in $\phi(x)$, $\psi(x)$, ω; Eq. (43)
ζ	= damping ratio
η	= ratio of modal frequency to frequency of first truncated mode
τ	= acoustic travel time, $(m\ell/ES)^{1/2}$
ϕ, Φ	= displacement mode shape
ψ	= stress mode shape
ω	= frequency

Subscripts

a	= points of applied force
b	= boundary points
c	= constrained body
f	= flexible body
i	= inertial
j,k	= indices
n	= a flexible-body mode
$\bar{n}$, N	= highest, all retained flexible-body mode(s)
Q	= all rigid-body and retained flexible-body modes
r, R	= one, all rigid-body mode(s)
ρ	= all residual modes

Superscripts

T	= matrix transpose
(0),(1),(2)	= order of approximation
$(\dot{\ })$, $(\ddot{\ })$	= first, second time derivative

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†Paired lower case, capital symbols denote scalar, vector or matrix quantities.

I. Introduction

THIS paper provides an improved means of representing the linearized dynamics of a structure based upon an incomplete set of normal modes of vibration. One of many possible applications is called *modal synthesis* or *component-mode synthesis*. These terms are employed generically to refer to any analytical process whereby the normal modes of a system of structural components are synthesized from the modes of the individual components. Hurty sparked an upsurge of publications in recent years on how to best conduct such a synthesis based upon a Rayleigh-Ritz procedure.[1] Often the end use of the system modes is the determination of forced response. Toward this end, another possible synthesis could be directed to forced system response without an intermediate step of determining system modes.

The focus here is on a new form of component-mode representation, not based upon a Rayleigh-Ritz procedure, which is believed to meet the various demands of practical applications in an exceptional manner. These demands can best be discussed in connection with a problem of synthesis. A fundamental goal of synthesis is to provide a reduced-size computational problem while maintaining the accuracy requirements. The process should facilitate the application of engineering judgment and empirical rules. In this respect, the use of normal coordinates for the component degrees of freedom is advantageous since it provides an excellent basis for the reduction via frequency discrimination. This is much better, for example, than the direct use of physical coordinates.

A second aspect of synthesis is the possibility that the components can be the portions of the overall structure assigned to different organizations. By minimizing the number of degrees of freedom required to represent each component, the synthesis approach reduces the amount of information to be exchanged and thereby simplifies the communication. Also, each organization is permitted to apply its own judgment for the reduction involving its own component.

A major advantage of synthesis is that modal testing, employing the same boundary conditions as the analysis, can be conducted on each individual component (perhaps by its own responsible organization) for corroboration of the analytically derived component modes. It is then possible to employ the data directly to adjust or replace the analytical modes. However, since any boundary conditions selected for the component normal modes will not be those occurring when the components act within a system, information in addition to the normal modes is usually required for a *complete* component representation. Ideally, the analytical method of representation should correspond to a practical capability for achieving a complete experimental representation. This aspect has not been emphasized as much as it should be. One practical consequence is that the more complete the experimental definition of the component dynamics, the less is the need for system-level testing.

Ideally, a component-mode representation will meet the following three goals: 1) efficiency in the sense that the required accuracy of the system dynamics is achieved with a minimum number of degrees of freedom, 2) total independence of the components for the analysis in that no knowledge of the remainder of the system is needed for the representation of any individual component, and 3) compatibility with practical test procedures in terms of the boundary conditions for the modal testing and in terms of the testing required for all the needed information beyond the normal modes.

The various published representations can be classified into four groups according to the boundary conditions employed:[‡] 1) Free-boundary modes, possibly with an account for the effects of modes not retained (termed "residual effects")[4-7]; 2) Fixed- boundary modes plus constraint "modes," the latter not being true normal modes[1-3]; 3) Boundary conditions based on characteristics of adjacent components or on arbitrary mass or spring restraints to "work" local structure[2,3,8-10]; 4) A mixture of free- and fixed-boundaries for the modes plus other "modes" (e.g., constraint or attachment modes) or "residual effects."[6,11]

Except for the use of free-boundary modes with residual effects, in one way or another most of these representations fail to meet the three goals we have set. The *classical method,* namely free-boundary modes without residual effects, has difficulty in representing local flexibility at boundary points; the result is weak convergence. The inclusion of residual effects corrects such difficulty. Representations which are based on a matching of the degree of constraint provided by adjacent components clearly do not meet the desire for independence of the component representations, and they also impose difficult requirements for testing. Complete correlation of test with analysis is not possible for methods which employ other than normal modes (for use in a Rayleigh-Ritz procedure); specifically, it has been concluded that mass coupling terms are not practically obtainable by test.[2,3] Actually, a complete test definition is not practically achievable for most methods, and so augmentation of the test data with theoretical information is necessary. For example, often a theoretical physical mass matrix is required. Deficiencies in the theoretical information combined with errors in the experimental data can lead to severe ill conditioning (e.g., imaginary natural frequencies).

While other methods achieve the reduction in the size of the system problem by reductions in the component representations, one method accomplishes the reduction after the system equations are developed.[12] The reduction is based upon a frequency-dependent approximation of the effect of a selected set of "reduced coordinates." System modes occurring in the general vicinity of the frequency employed for the approximation are thereby improved in accuracy. This type of approach fails to meet the objectives set forth here in that it does not 2inimize the numer of component degrees of freedom which must be calculated, communicated, and/or experimentally determined.

It is contended here that the methods which have the best potential to meet the aforementioned goals are those employing free-boundary modes plus residual effects. The MacNeal method for free boundaries and the method of this paper fall into this category.[4-6] MacNeal put forth a method which accounts for the static contribution of neglected modes (called residual flexibility), and the result is vastly more efficient than the classical method which completely ignores the neglected modes. Inexplicably, the MacNeal method has not met with wide acceptance. Perhaps the reason is that MacNeal uses impedance/admittance terminology which has caused some to incorrectly conclude that the method is directed toward analog computation.[3] Or, perhaps there has been a strong unwillingness to depart from the familiar Rayleigh-Ritz approach. We shall here avoid terminology which is unfamiliar to most structural dynamicists, and we shall attempt to provide a strong intuitive basis to enhance receptiveness to this new method.

The method of this paper is an extension of the MacNeal method in that an inertial, and possibly a dissipative, contribution to the residual effects is also included. This residual inertia is not MacNeal's "residual mass," which he and others employ when boundary points are fixed for the modal analysis.[6,13] As we shall discuss later, the free-boundary residual effects can be fully determined by augmenting modal testing with certain frequency-response measurements. The new method meets our stated independence and test goals, and it will be shown to yield exceptional convergence.

The efficiency of the new method for a carefully chosen test problem will be evaluated by comparison to four other methods: classical, MacNeal, Hurty, and Bamford. The classical, MacNeal, and new methods form a logical progression in terms of the approximation of free-free modes not retained. The Hurty method is included because studies comparing the various methods (other than MacNeal's) have led to the conclusion that the Hurty method (or its derivatives) tend to yield the best convergence when there are not a relatively large number of interconnection coordinates.[3] What is called the Bamford method here is the use of attachment modes to augment free-boundary modes, which is actually an extrapolation of Bamford's application; it is included because it employs the same displacement functions as does the MacNeal method, but with a Rayleigh-Ritz approach.[11]

II. Fundamentals of Concept

A. Approximation for Individual Mode

We begin with the differential equation for the generalized displacement $q_n(t)$ in a viscously-damped flexible-body mode in either of the following basic forms

$$m_n \ddot{q}_n(t) + c_n \dot{q}_n(t) + k_n q_n(t) = f_n(t) \tag{1a}$$

$$m_n [\ddot{q}_n(t) + 2\zeta_n \omega_n \dot{q}_n(t) + \omega^2_n q_n(t)] = f_n(t) \tag{1b}$$

A first-order low-frequency approximation $q_n^{(1)}$ can be obtained by neglecting the inertial and damping terms, yielding a wholly *static displacement at each instant of time*

$$k_n q_n^{(1)}(t) = f_n(t) \tag{2}$$

‡A more complete list of references and a detailed comparison of various methods is found in Refs. 2 and 3.

A second-order approximation $q_n^{(2)}$ can be obtained by using $q_n^{(1)}$ to approximate the inertial and damping terms

$$k_n q_n^{(2)}(t) = f_n(t) - \ddot{f}_n(t)/\omega_n^2 - 2\zeta_n \dot{f}_n(t)/\omega_n \quad (3)$$

This iteration process could continue indefinitely, but this is sufficient for our purposes.§

We now assess the error incurred at low frequencies with these two orders of approximation by writing the two associated error displacements, $\tilde{q}_n^{(1)}(t) = q_n(t) - q_n^{(1)}(t)$ and $\tilde{q}_n^{(2)}(t) = q_n(t) - q_n^{(2)}(t)$, in terms of the exact displacement under steady-state sinusoidal conditions at the frequency ω

$$\tilde{q}_n^{(1)}(\omega) = (\omega^2/\omega_n^2 - i2\zeta_n\omega/\omega_n) q_n(\omega) \quad (4a)$$

$$\tilde{q}_n^{(2)}(\omega) = (\omega^2/\omega_n^2 - i2\zeta_n\omega/\omega_n)^2 q_n(\omega) \quad (4b)$$

Note that for $\zeta \ll \omega/\omega_n$ the fractional errors become ω^2/ω_n^2 and ω^4/ω_n^4 in the two instances. Plots of the normalized q_n, $q_n^{(1)}$, $q_n^{(2)}$ vs ω/ω_n for zero damping are shown in Fig. 1. The method due to MacNeal employs the static approximation $q_n^{(1)}$ for the residual modes. This results in a vast improvement in accuracy over the classical approach which completely ignores modes not to be retained in the representation.[4-6,14,15] The method of this paper goes one step beyond MacNeal and employs the second-order approximation for the residual modes, thereby achieving additional accuracy.

1. Example 1

The free rod shown in Fig. 2 will serve as the sample system throughout the paper. In this first example we write the frequency response of base acceleration per unit base force as a summation of modal contributions

$$[m\ddot{u}(0)/f] = 1 + \sum_{j=1}^{\infty} \frac{2\omega^2\tau^2}{\omega^2\tau^2 - j^2\pi^2} \quad (5a)$$

We shall now use the two orders of approximation for all those modes above $j = \bar{n}$. The results can be written in the form

$$[m\ddot{u}^{(1)}(0)/f] = 1 - \frac{\omega^2\tau^2}{3} + \sum_{j=1}^{\bar{n}} \frac{2\omega^4\tau^4}{j^2\pi^2(\omega^2\tau^2 - j^2\pi^2)} \quad (5b)$$

$$[m\ddot{u}^{(2)}(0)/f] = 1 - \frac{\omega^2\tau^2}{3} - \frac{\omega^4\tau^4}{45} + \sum_{j=1}^{\bar{n}} \frac{2\omega^6\tau^6}{j^4\pi^4(\omega^2\tau^2 - j^2\pi^2)} \quad (5c)$$

In each case the first term 1 is the contribution of the rigid-body mode; the second term, $-\omega^2\tau^2/3$, comes from the infinite sum of the first-order approximations for all the modes. The next term in Eq. (5c), $-\omega^4\tau^4/45$, is the additional contribution from the second-order approximations for all the modes. The summation terms contain the leftover contributions of the first $\bar{n}$ flexible-body modes so that they contribute without approximation. The exact response is known to be $\omega\tau/\tan\omega\tau$; we note that the terms ahead of the summation in Eqs. (5a-c) are those of the Maclaurin-series expansion for this exact response.

We now observe the rate of convergence to the exact result

§We elect to stop at the second-order approximation since to go beyond this level would result, for a component, in a system of equations of higher than the second order.

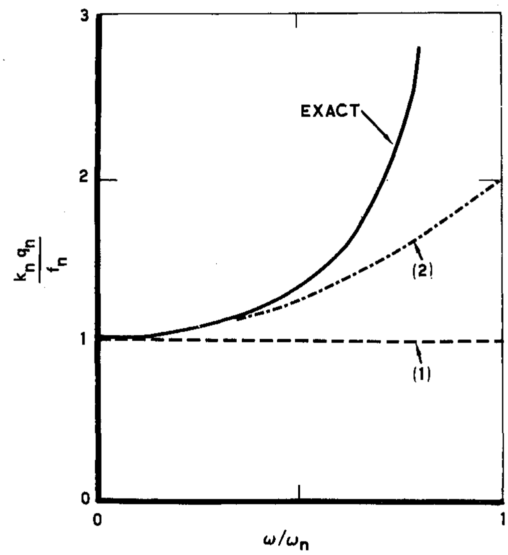


Fig. 1 Comparison of exact and approximate generalized displacement for zero damping.

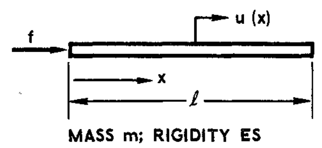


Fig. 2 Free rod.

of zero for $\omega\tau = \pi/2$, including first a zero th-order (classical) approximation which totally neglects the higher modes

$$[m\ddot{u}^{(0)}(0)/f] = 1, 0.333, 0.200, 0.143, 0.111, \ldots 0.034$$
$$\text{for } \bar{n} = 0, 1, 2, 3, 4, \ldots 14 \quad (6a)$$

$$[m\ddot{u}^{(1)}(0)/f] = 0.178, 0.011, 0.0025, 0.0010$$
$$\text{for } \bar{n} = 0, 1, 2, 3 \quad (6b)$$

$$[m\ddot{u}^{(2)}(0)/f] = 0.042, 0.00058, 0.000055, 0.000011$$
$$\text{for } \bar{n} = 0, 1, 2, 3 \quad (6c)$$

Observe that the classical approach displays poor convergence. The progressive improvement in convergence with order of approximation is evident.

B. Relationship to Mode-Acceleration Method

A replacement of the displacement $q_n(t)$ in Eq. (1b) by $q_n^{(1)} + \tilde{q}_n^{(1)}$ or $q_n^{(2)} + \tilde{q}_n^{(2)}$ yields, respectively

$$\ddot{q}_n(t) + 2\zeta_n\omega_n\dot{q}_n(t) + \omega_n^2\tilde{q}_n^{(1)}(t) = 0 \quad (7a)$$

$$\ddot{\tilde{q}}_n^{(1)}(t) + 2\zeta_n\omega_n\dot{\tilde{q}}_n^{(1)}(t) + \omega_n^2\tilde{q}_n^{(2)}(t) = 0 \quad (7b)$$

Equation (7a) shows that the elastic force due to the first-order error displacement $\tilde{q}_n^{(1)}$ equilibrates the total inertial and damping forces in a mode, *even during forced oscillation.* Likewise, Eq. (7b) shows that the elastic force due to $\tilde{q}_n^{(2)}$ equilibrates the inertial and damping forces due to the first-order error displacement $\tilde{q}_n^{(1)}$. The consequences of both equilibrations are now discussed with relation to the mode-acceleration method.[16]

In the *mode-displacement method* the stress σ at a point x is found by a summation over the incomplete set of flexible-body normal modes

$$\sigma(x) = \sum_{j=1}^{\bar{n}} \psi_{nj}(x) q_{nj} \quad (8)$$

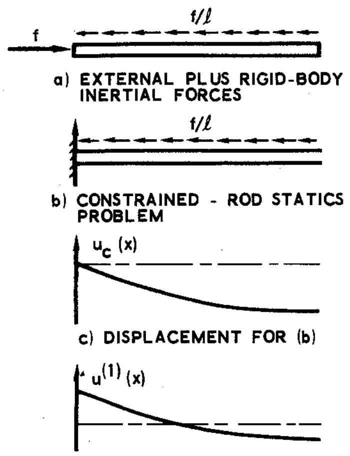


Fig. 3 First-order statics problem for rod.

where $\psi_{nj}(x)$ is the stress at x due to a unit generalized displacement in the jth mode. This method corresponds to the classical method wherein there is no account for modes not retained in the analysis. An alternative method, which has been shown to improve the convergence, is the mode-acceleration method; stresses are determined from an equilibrium with the applied forces and the inertial forces resulting from the rigid-body and flexible-body modal accelerations. For an undamped body, the result is expressed in the form[16]

$$\sigma(x) = \sigma(x)_{\text{static}} - \sum_{j=1}^{\bar{n}} [\psi_{nj}(x)/\omega_{nj}^2]\ddot{q}_{nj} \tag{9}$$

The "static stress" includes the effect of external forces and inertial forces associated with rigid-body accelerations; it is simply the stress ordinarily computed when the body is assumed rigid. The quantity $-\psi_{nj}^2$ is the stress at x due to a unit generalized acceleration in the jth mode.

If we now formulate what can be termed a *modified mode-displacement method* based on the first-order approximation of residual modes, we obtain (for any damping)

$$\sigma(x) = \sigma(x)_{\text{static}} + \sum_{j=1}^{\bar{n}} \psi_{nj}(x)\tilde{q}_{nj}^{(1)} \tag{10}$$

The "static stress" is that obtained from the statics problem for the overall flexible body and is identical to that in the mode-acceleration method since the forces acting on the body are the same. Since all modes are accounted for statically, for each mode to be represented completely we need only add stresses due to the excess-over-static displacement, $\tilde{q}_n^{(1)}$. For undamped vibration in a mode, we have directly from Eq. (7a)

$$\tilde{q}_{nj}^{(1)} = -\ddot{q}_{nj}/\omega_{nj}^2$$

which, if inserted into Eq. (10), produces an expression for stress which is identical to that given by Eq. (9) for the mode-acceleration method. This identity holds as well in the presence of damping, which can be treated in the mode-acceleration method by replacing $\ddot{q}_{nj}$ with $\ddot{q}_{nj} + 2\zeta_{nj}\omega_{nj}\dot{q}_{nj}$. *It is concluded that for prescribed external forces, the improvement in deflections provided by a residual static flexibility leads by a displacement type of method to stresses which are identical to those given by the mode-acceleration method.* This is a direct consequence of the equilibrium of forces expressed by Eq. (7a).

As a consequence of Eq. (7b) it can be shown that the *stresses resulting from displacements based on the second-order approximation for residual modes are identical with stresses which would result from the application of the mode-acceleration method to the deflections based on first-order residual effects.* Thus the second-order approximation produces stresses which can also be obtained as follows: 1) obtain displacement due to the prescribed forces classically, 2) obtain stresses by mode-acceleration, 3) obtain new displacements which correspond to these stresses (i.e., through use of stress-strain relationships), and 4) apply mode acceleration to the new displacements. In general, each new level of approximation of residual effects brings about the same improvement in stresses (over the previous level) as would the application of mode-acceleration.

III. Exposition of Method

A. Approximation for Overall Flexible-Body

For practicality, it is necessary to be able to accomplish the approximation process without knowledge of the individual residual modes. This can be achieved as follows: 1) solve for the entire flexible-body displacement (that is, the summation of displacements in the complete set of flexible-body modes) due to the external forces, and 2) remove the contribution of the modes to be retained, thereby leaving as a remainder the contribution of the entire group of residual modes. The first aspect is covered in this section.

1. Statics problem

First we identify the statics problem from which the first-order approximate deflection is determined. If the body possesses no rigid-body modes, we have nothing more than a usual statics problem for determination of the deflection to any applied force. Of special interest is the case in which the body is a free one. We wish to somehow prevent the body from undergoing rigid-body motion, yet not effect its flexible-body behavior. This is accomplished by the following steps. (An illustration of each step is provided by Fig. 3 which deals with the axial vibration of a uniform free rod due to an end force.)

a) Superimpose on the set of external forces the inertial forces which would result from rigid-body accelerations. These inertial forces do not contribute to the generalized force in any flexible-body mode in view of the orthogonality conditions on the modes. Hence, the resulting displacement in each flexible-body mode is unaffected.

b) To provide a statics problem involving finite flexibilities, impose any determinate set of constraints to prevent rigid-body displacement. Since the overall set of forces is in equilibrium, the forces of constraint will be zero.

c) Solve for the displacements.

d) Adjust the displacements from Step c to remove any rigid-body contributions. The adjustment involves adding arbitrary rigid-body displacements and then orthogonalizing the results with respect to each rigid-body mode. The result is the complete displacement for the first-order statics problem [that is, the sum of the $q_n^{(1)}$].

The second-order statics problem is then solved by including in Step a the mass and damping forces due to the first-order flexible-body deflection, comparable to what was done in Eq. (3). The remainder of the process is carried out as before.

2. Example 2

For the rod problem, the first-order (static) displacement shown in Fig. 3d is

$$u_f^{(1)}(x) = \frac{f\ell}{ES}\left[\frac{1}{3} - \frac{x}{\ell} + \frac{x^2}{2\ell^2}\right] \tag{11}$$

The external force in Step a for the second-order statics problem is then

$$f^{(2)}(x) = -\frac{f}{\ell}\left[1 + s^2\tau^2\left(\frac{1}{3} - \frac{x}{\ell} + \frac{x^2}{2\ell^2}\right)\right] \tag{12}$$

where the Laplace variable "s" has been introduced. The resulting deflection of the constrained rod in Step c is

$$u_c^{(2)}(x) = -\frac{f\ell}{ES}\left[\frac{x^2}{2\ell^2} - \frac{x}{\ell} + \frac{s^2\tau^2}{6}\,\frac{x^2}{\ell^2}\left[1 - \frac{x}{\ell} + \frac{x^2}{4\ell^2}\right]\right] \tag{13}$$

The final result in Step d is

$$u_f^{(2)}(x) = \frac{f\ell}{ES}\left[\frac{1}{3} - \frac{x}{\ell} + \frac{x^2}{2\ell^2} - \frac{s^2\tau^2}{3}\left[\frac{1}{15} - \frac{x^2}{2\ell^2} + \frac{x^3}{2\ell^3} - \frac{x^4}{8\ell^4}\right]\right] \tag{14}$$

which is the summation of all the $q_n^{(2)}$ contributions. For $s=0$, Eqs. (12-14) become those of the first-order statics problem.

3. Matrix formulation of statics problem.

We now provide a matrix formulation of this statical process (both first- and second-order problems) for a discretized system.¶ The displacements of the physical coordinates due to generalized rigid-body displacements are

$$U_R = \Phi_R Q_R \tag{15}$$

and the differential equation for the generalized rigid-body displacements due to an array of external forces F is

$$M_R \ddot{Q}_R = \Phi_R^T F \tag{16}$$

where $M_R = \Phi_R^T M \Phi_R$ is the rigid-body generalized mass matrix and M is the physical mass matrix. The combined set of external plus inertial forces (F and F_i, respectively) is then written as

$$F + F_i = F - M\ddot{U}_R = AF, \quad A = I - M\Phi_R M_R^{-1}\Phi_R^T \tag{17}$$

The relationship of the combined forces $F+F_i$ to the external forces F is constrained by as many equilibrium conditions as there are rigid-body degrees of freedom $\bar{r}$. Hence when $\bar{r}\neq 0$, A is singular with rank $\bar{m}-\bar{r}$.[17] In the special case where there are no rigid-body modes ($\bar{r}=0$), A is simply the unit matrix I because no rigid-body inertial forces exist.

With the flexibility matrix of the constrained body denoted by G_c, the corresponding deflection for this first-order problem is

$$U_c^{(1)} = G_c(F+F_i) = G_c AF \tag{18}$$

This is an unusual statics problem in that the resulting flexibility matrix G_c is singular and in general nonsymmetric. For example, any coordinate which is fixed to prevent rigid-body motion will produce a corresponding null row in G_c. Since $\bar{r}$ conditions have been imposed on the body to prevent that number of rigid-body deflections, G_c and G_cA are singular with rank $\bar{m}-\bar{r}$. Were it not for the fact that the body had been constrained, G_c and G_cA would not exist at all.

The constrained deflection, $U_c^{(1)}$, differs from the desired first-order (or static) flexible-body displacement, $U_f^{(1)}$, by certain generalized rigid-body displacements Q_R'; that is

$$U_f^{(1)} = U_c^{(1)} + \Phi_R Q_R' \tag{19}$$

The requirement that $U_f^{(1)}$ be free of any rigid-body contribution is satisfied if we determine the Q_R' such that $U_f^{(1)}$ is orthogonal to the rigid-body modes; namely

$$\Phi_R^T M[U_c^{(1)} + \Phi_R Q_R'] = 0 \tag{20}$$

Solving Eq. (20) for Q_R', and employing $M_R = \Phi_R^T M \Phi_R$, we substitute into Eq. (19) to obtain the free-body *symmetric*-flexibility matrix, G, for the first-order approximation**

$$U_f^{(1)} = GF, \quad G = A^T G_c A \tag{21}$$

Recognize that $U_f^{(1)}$ is the same as the result of combining the first-order generalized displacements in the complete set of flexible-body modes. The fixity constraints placed on the body to enable the determination of G_c have, in effect, been eliminated with the result that G is symmetric. However, G is still singular with rank $\bar{m}-\bar{r}$.

The second-order approximation is found by adding to the external forces the inertial and damping forces resulting from the first-order deflections. That is, the second-order version of Eq. (18) yields the constrained deflection

$$U_c^{(2)} = G_c A[F - M\ddot{U}_f^{(1)} - C\dot{U}_f^{(1)}] \tag{22}$$

As before, rigid-body contributions are removed to yield

$$U_f^{(2)} = GF - H\ddot{F} - B\dot{F}, \quad H = GMG, \quad B = GCG \tag{23}$$

where G is given in Eq. (21). Thus we see that the correction terms brought about by the second-order approximation involve simply the physical mass and damping matrices pre- and post-multiplied by the flexibility matrix G found from the first-order problem. Since G, M, and C are symmetric, so are H and B.

It is recognized that, in general, an analytical derivation of the damping matrix C is not within the state of the art. However, C might still be employed to represent certain known damping characteristics, such as that due to concentrated damping devices.

B. Contribution of Residual Modes

Since we have determined the approximate deflection for the overall flexible body, we remove the contribution of the modes to be retained, leaving as a remainder the approximation for all the residual modes. Thus, if $U_{f\rho}^{(1)}$ denotes the first-order residual deflection of the flexible body, we can write directly

$$U_{f\rho}^{(1)} = G_\rho F, \quad G_\rho = G - G_N, \quad G_N = \Phi_N K_N^{-1}\Phi_N^T \tag{24}$$

where Φ_N and K_N are the mode-shape matrix and the generalized stiffness matrix, respectively, for the group of retained modes.

In a like manner, the second-order residual contribution is given by

$$U_{f\rho}^{(2)} = G_\rho F - H_\rho \ddot{F} - B_\rho \dot{F},$$
$$H_\rho = H - H_N, \quad H_N = G_N M G_N \tag{25a}$$
$$B_\rho = B - B_N, \quad B_N = G_N C G_N$$

where H and B are given in Eq. (23). Making use of modal orthogonality (e.g., $G_N M G_\rho = 0$) we can also write

$$H_\rho = G_\rho M G_\rho, \quad B_\rho = G_\rho C G_\rho \tag{25b}$$

The latter is true only if the modal transformation diagonalizes the damping matrix.

We have, therefore, accomplished the objective of deter-

¶It is suggested that the reader perform the analysis of this section on a two-mass, one-spring system (m_1, k, m_2) to help crystallize the concept.

**Identical to MacNeal's result in Eq. (46) of Ref. 6.

mining the residual contribution of modes not retained without knowledge of those modes.

To recapitulate: a) The flexibility matrix G is determined from a statics problem which requires knowledge of the rigid-body modes; see Eq. (21). b) The inertial and damping correction matrices in the second-order approximation, H and B respectively, are simply a transformation of the mass and damping matrices by pre- and post-multiplication with G; see Eq. (23). c) The corresponding residual matrices G_ρ, H_ρ, B_ρ are found from the overall matrices G, H, B by removal of the contribution of the modes to be retained; see Eqs. (24) and (25).

C. Stiffness-Matrix Formulation

A useful implementation of the representation is a stiffness-matrix form for the equations of motion of the structural component with residual effects included. Laplace notation is employed. Let F_a, a subset of F, represent the applied forces. We begin by relating the corresponding displacements U_a to the generalized displacements of the retained modes Q and to the F_a

$$U_a = \Phi_a Q + (G_{\rho a} - H_{\rho a}s^2 - B_{\rho a}s)F_a \tag{26}$$

The first term yields the contribution of all retained modes (including rigid-body modes) and the remainder is the second-order residual contribution from the neglected modes according to Eq. (25a). Next we solve Eq. (26) for the applied forces under the assumption that the residual flexibility contribution is the dominant residual effect (i.e., $G_{\rho a} \gg H_{\rho a}s^2 + B_{\rho a}s$) and also that $G_{\rho a}^{-1}$ exists. The result is

$$F_a \cong (K_{\rho a} + M_{\rho a}s^2 + C_{\rho a}s)\ (U_a - \Phi_a Q) \tag{27}$$

where

$$K_{\rho a} = G_{\rho a}^{-1}, \quad M_{\rho a} = G_{\rho a}^{-1}\bar{H}_{\rho a}G_{\rho a}^{-1}, \quad C_{\rho a} = G_{\rho a}^{-1}B_{\rho a}G_{\rho a}^{-1}$$

and

$$\bar{H}_{\rho a} = H_{\rho a} + B_{\rho a}G_{\rho a}^{-1}B_{\rho a}$$

To obtain this result, it was necessary to employ the following Maclaurin-series expansion up to order s^2

$$[I - G_{\rho a}^{-1}(H_{\rho a}s^2 + B_{\rho a}s)]^{-1} \cong I + G_{\rho a}^{-1}(\bar{H}_{\rho a}s^2 + B_{\rho a}s) \tag{28}$$

In accordance with the prior discussion of the rank of G following Eq. (21), the existence of the inverse of $G_{\rho a}$ requires that its order (equal to the number of applied forces) be no greater than the number of independent degrees of freedom for the statics problem (i.e., $\bar{m} - \bar{r}$).

In addition, we require the equations of motion for all the retained modes (flexible- and rigid-body modes, denoted jointly by the subscript "Q")

$$(M_Q s^2 + C_Q s + K_Q)Q = \Phi_a^T F_a \tag{29}$$

where Φ_a contains the modal displacements corresponding to the applied forces F_a. From Eqs. (27) and (29) we obtain the desired *stiffness-matrix formulation*

$$\begin{bmatrix} Z_{QQ} & Z_{Qa} \\ Z_{Qa}^T & Z_{aa} \end{bmatrix} \begin{Bmatrix} Q \\ U_a \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_a \end{Bmatrix} \tag{30}$$

where

$$Z_{QQ} = M_Q s^2 + C_Q s + K_Q + \Phi_a^T Z_{aa}\Phi_a$$
$$Z_{aa} = M_{\rho a}s^2 + C_{\rho a}s + K_{\rho a}$$
$$Z_{Qa} = -\Phi_a^T Z_{aa}$$

If only the first-order residual contribution is employed as was done by MacNeal (i.e., $M_{\rho a} = C_{\rho a} = 0$), the formulation becomes that of MacNeal for free-boundary modes [see Eq. (47) of Ref. 6]. It is not essential that this stiffness form be employed, but it will usually be convenient to do so. An alternative is a combination of Eqs. (26) and (29).

D. Physical Displacements

The solution of a problem by use of Eq. (30), or a combination of Eqs. (26) and (29), will yield for each component the generalized displacements Q, the subset of physical displacements U_a, and the corresponding applied forces F_a. The complete set of physical displacements U for each component is then found using

$$U = \Phi Q + (G_\rho - H_\rho s^2 - B_\rho s)F \tag{31}$$

which is based on Eq. (25a). The complete force vector F contains nonzero elements only for the applied forces F_a. Use of Eq. (31) implies a first and second differentiation of forces, so it is essential that its use does not extend into the high-frequency range beyond which the residual approximations are valid. This presents no difficulty for solutions in the frequency domain (e.g., a frequency-response determination or a modal-synthesis problem). In the time domain, however, an actual differentiation of the forces must be preceded by an adequate removal of high-frequency content within those forces. An alternative is to neglect the residual inertial and damping effects.

If the stiffness formulation of Eq. (30) is employed for a problem, the U_a obtained from Eq. (31) will be somewhat different from those previously found directly from the system equations; they will be the same if a combination of Eqs. (26) and (29) is employed. This discrepancy arises because the approximation in Eq. (28) is used for the stiffness formulation.[††] As long as the assumptions leading to Eq. (28) are valid, each set of U_a is valid to the order s^2, which is the same as the order of validity for the basic method. *We therefore expect that the magnitude of the discrepancy in the U_a, arising from a stiffness formulation, may be a useful indicator of the general level of displacement uncertainty to be expected by use of the method.* Such an indication is illustrated in the sample problem of the Sec. III E.

If a major inconsistency should arise in the displacement of any applied force, alternatives are to employ only the first-order approximation for that point or to refrain from using the stiffness formulation. In an extreme case, the stiffness formulation may not be usable because of ill conditioning of $G_{\rho a}$.

E. Application to Modal Synthesis

As an example of an application of the component-mode representation of this paper, we review briefly its use for modal synthesis. In this application the applied forces for a component only occur at the boundaries (that is, interfaces with other components). These forces and corresponding displacements are designated by F_b and U_b, respectively. The formulation of a modal-synthesis problem can be accomplished in a standard manner with components individually represented by the stiffness formulation of Eq. (30). System eigensolutions are then extracted: s_j^* and corresponding eigenvectors Q_j^*, U_{bj}^*. The component boundary forces will not appear explicitly in the system equations since they are internal forces of the system; the value of these forces in the system mode, F_{bj}^*, can be found by application of Eq. (27)

$$F_{bj}^* = (K_{\rho b} + M_{\rho b}s_j^{*2} + C_{\rho b}s_j^*)\ (U_{bj}^* - \Phi_b Q_j^*) \tag{32}$$

††A basis for U_a consistent with Eq. (30) would be to determine the complete set of displacements by use of a generalization of Eq. (27) to make it applicable to all points on a component. We choose not to do so because of ill conditioning.

$$U_j^* = \Phi Q_j^* + (G_\rho - H_\rho s_j^{*2} - B_\rho s_j^*) F_j^* \tag{33}$$

where F_j^* contains nonzero elements only for the boundary forces F_{bj}^*.

IV. Demonstration and Evaluation for Cantilevered Rod

We shall now demonstrate the method of this paper on a carefully chosen system and compare its accuracy relative to that of other selected methods. The test problem is to develop the normal modes for an undamped uniform cantilevered rod. The rod is shown in Fig. 2 and is to be fixed at the base ($x=0$). The problem is believed to be an ideal test case for evaluation of the basic modal-synthesis capability for our purposes. Its virtues are 1) The availability of simple closed-form solutions. 2) The presence of an infinity of normal modes tending to become relatively more closely spaced with increasing order. 3) The lack of emphasis or de-emphasis of local dynamic properties at the boundaries. 4) The severity of the test, but with equality for a method based upon either free or fixed boundaries at both ends of the rod. First we will describe the application of the new method and the other methods applied for comparative purposes. Then we will show selected results and compare the accuracy of the various methods.

A. Description of Methods

1. New method

From Example 2, the flexibility $g(x)$ and the inertial contribution to the dynamic flexibility $h(x)$ are seen from Eq. (14) to be

$$g(x) = \frac{\ell}{ES}\left[\frac{1}{3} - \frac{x}{\ell} + \frac{x^2}{2\ell^2}\right] \tag{34a}$$

$$h(x) = \frac{\ell\tau^2}{3ES}\left[\frac{1}{15} - \frac{x^2}{2\ell^2} + \frac{x^3}{2\ell^3} - \frac{x^4}{8\ell^4}\right] \tag{34b}$$

The flexible-body modes are defined by

$$\omega_{nj} = j\pi/\tau,\ \phi_{nj}(x) = \cos j\pi x/\ell,\ m_{nj} = m/2 \tag{35}$$

and the rigid-body mode by $\phi_r(x) = 1$, $m_r = m$. Based on Eq. (3), the second-order contribution to the deflection by the j^{th} mode is

$$u_{nj}^{(2)}(s) = \frac{2\ell\cos j\pi x/\ell}{j^2\pi^2 ES}\left[1 - \frac{s^2\tau^2}{j^2\pi^2}\right] f(s) \tag{36}$$

Using these results, the residual flexibility for all those modes beyond $j=\bar{n}$ is

$$g_\rho(x) = \frac{\ell}{ES}\left[\frac{1}{3} - \frac{x}{\ell} + \frac{x^2}{2\ell^2} - \frac{2}{\pi^2}\sum_{j=1}^{\bar{n}}\frac{1}{j^2}\cos\frac{j\pi x}{\ell}\right] \tag{37a}$$

and the corresponding residual inertial effect is

$$h_\rho(x) = \frac{\ell\tau^2}{3ES}\left[\frac{1}{15} - \frac{x^2}{2\ell^2} + \frac{x^3}{2\ell^3} - \frac{x^4}{8\ell^4} - \frac{2}{\pi^4}\sum_{j=1}^{\bar{n}}\frac{1}{j^4}\cos\frac{j\pi x}{\ell}\right] \tag{37b}$$

From Eq. (27), the residual stiffness and mass at the base ($x=0$) are given by

$$k_{\rho b} = \frac{ES}{\ell}\left[\frac{1}{3} - \frac{2}{\pi^2}\sum_{j=1}^{\bar{n}}\frac{1}{j^2}\right]^{-1}$$

$$m_{\rho b} = \frac{m}{3}\left[\frac{1}{15} - \frac{2}{\pi^4}\sum_{j=1}^{\bar{n}}\frac{1}{j^4}\right] \times \left[\frac{1}{3} - \frac{2}{\pi^2}\sum_{j=1}^{\bar{n}}\frac{1}{j^2}\right]^{-2} \tag{38}$$

Since $u_b=0$, the stiffness formulation of Eq. (30) (and also the eigenvalue problem) becomes simply

$$[(M_Q + \Phi_b^T M_{\rho b}\Phi_b)s^2 + (K_Q + \Phi_b^T K_{\rho b}\Phi_b)]Q = 0 \tag{39}$$

Since each element of Φ_b is unity, the expanded form of the eigenvalue program is

$$\left[\begin{bmatrix} m+m_{\rho b} & m_{\rho b} & \cdots & m_{\rho b} \\ m_{\rho b} & (m/2)+m_{\rho b} & \cdots & m_{\rho b} \\ \vdots & \vdots & \vdots & \vdots \\ m_{\rho b} & m_{\rho b} & \cdots & (m/2)+m_{\rho b} \end{bmatrix} s^2 + \begin{bmatrix} k_{\rho b} & k_{\rho b} & \cdots & k_{\rho b} \\ k_{\rho b} & (\pi^2 m/2\tau^2)+k_{\rho b} & \cdots & k_{\rho b} \\ \vdots & \vdots & \vdots & \vdots \\ k_{\rho b} & k_{\rho b} & \cdots & (\bar{n}^2\pi^2 m/2\tau^2)+k_{\rho b} \end{bmatrix}\right] \begin{Bmatrix} q_r \\ q_1 \\ \vdots \\ q_{\bar{n}} \end{Bmatrix} = 0 \tag{40}$$

The eigenvalues $s_{nk}^* = i\omega_{nk}^*$ and corresponding eigenvectors

$$(q_{rk}^*, q_{1k}^*, \ldots, q_{jk}^*, \ldots, q_{\bar{n}k}^*)$$

are then determined for $k=1,2,\ldots\bar{n}+1$. The physical mode shapes, ϕ_{nk}^*, are based on Eq. (31). Making use of the equation of motion for the rigid-body mode (i.e., $f_k^* = ms^2 q_{rk}^*$), the result is

$$\phi_{nk}^*(x) = \alpha\{[1 - m\omega_{nk}^{*2}(g_\rho(x) + h_\rho(x)\omega_{nk}^{*2})]q_{rk}^* + \sum_{j=1}^{\bar{n}} q_{jk}^* \cos j\pi x/\ell\} \tag{41}$$

where α is a normalizing factor to provide a generalized mass equal to the physical mass, m. Finally, the stress mode shapes are found from a differentiation of the displacement mode shapes

$$\psi_{nk}^*(x) = -E[\mathrm{d}\phi_{nk}^*(x)/\mathrm{d}x] \tag{42}$$

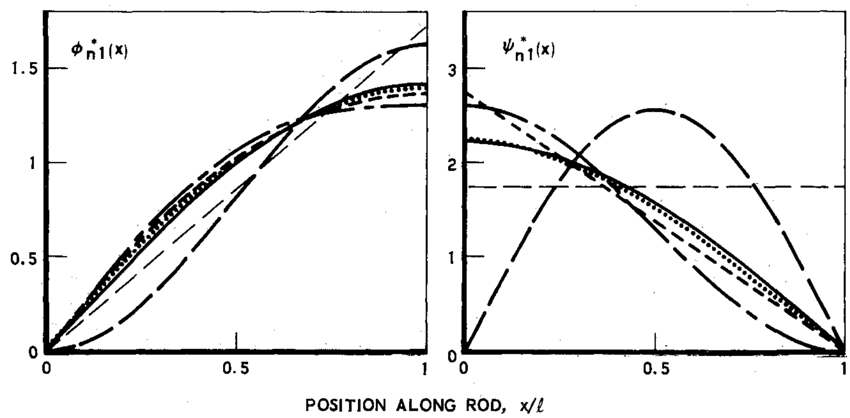
As an example of the new method we show partial results for the case $\bar{n}=0$ (that is, only the rigid-body mode is retained and all the flexible-body modes are treated in a residual fashion). From Eq. (38), $k_{\rho b} = 3ES/\ell$, $m_{\rho b} = m/5$ and the eigenvalue problem of Eq. (40) becomes simply

$$[(6/5)ms^2 + 3ES/\ell]q_r = 0$$

yielding a nondimensional natural frequency for the first cantilevered mode of $\omega^*\tau = (5/2)^{1/2} = 1.5811\ldots$ compared to an exact value of $\pi/2 = 1.5708\ldots$; the fractional error is only 0.0066 with no flexible-body normal modes having been employed!

2. MacNeal method

The MacNeal method is a special case of the new method where the residual inertial effects are not included. Results



	$\omega^*_{n1}\tau$	ϵ_ω	ϵ_ϕ	ϵ_ψ
EXACT	1.571	0	0	0
NEW	1.581	0.0066	0.013	0.025
MACNEAL	1.732	0.10	0.027	0.036
BAMFORD	1.581	0.0066		
HURTY	1.732	0.10	0.085	0.34
HURTY VAR	1.599	0.018	0.047	0.14
CLASSICAL	1.814	0.15	0.14	0.43

Fig. 4 Results for first cantilever mode based on first-order eigenvalue problem.

can be obtained by setting to zero the quantities $h(x)$, $h_p(x)$, m_{pb}. Note that the stiffness matrix will be full, while the mass matrix is diagonal.

For the $n=0$ case, the eigenvalue problem is

$$(ms^2+3\ ES/\ell)q_r=0$$

yielding a nondimensional natural frequency of $\omega^*\tau=3^{1/2}$ with a fractional error of 0.10; this compares to 0.0066 for the new method.

3. Classical Method

The classical method uses the rigid-body and first $\bar{n}$ free-free modes for the displacement functions. The eigenvalue problem is then developed by imposing the constraint of zero deflection at the base. There results a diagonal stiffness matrix and a full mass matrix.

4. Hurty Method and a Variation

The Hurty method employs for the displacement functions the rigid-body mode shape and the shape $1\text{-}x/\ell$ for the constraint modes, along with the desired number of fixed-fixed mode shapes. Kinetic and strain energies are developed using these functions, and are inserted into the Lagrange equations to formulate a corresponding mass and stiffness matrix. This cantilevered-rod problem is an unusual one for the Hurty method in that an attachment point is to be freed. Based upon discussion with Hurty this problem could be viewed as one having a spring on the end to be freed, and then letting the spring stiffness go to zero. The result is the absence of any boundary condition at the free end (i.e., a zero stress condition at the free end is not imposed). The generated eigenproblem involves a diagonal stiffness matrix and a mass matrix which is diagonal except for the row and column corresponding to the $1\text{-}x/\ell$ mode shape. For a variation of the Hurty method, we include the zero strain condition at the free end, thus allowing an additional fixed-fixed mode to be employed to develop a given order eigenproblem. This variation produces an eigenproblem wherein both the stiffness and mass matrices are full.

5. Bamford method

The Bamford method, as identified here, employs an attachment mode in addition to the rigid-body mode and the free-free flexible-body modes. This goes beyond Bamford's actual application of attachment modes as he dealt with them only for constrained bodies. This attachment mode is nothing more than the previously discussed static displacement shape for a base force [i.e., the $g(x)$ in Eq. (34)]. As with the Hurty method, energies are developed and mass and stiffness matrices are then derived. The imposition of the zero-deflection constraint at the base then leads to the eigenvalue problem on basically full stiffness and mass matrices.

B. Results

The following error measures are defined: ϵ_ω is the fractional error in the natural frequency; ϵ_ϕ and ϵ_ψ are the root-mean-square (rms) errors in the displacement and stress mode shapes, respectively, as a fraction of their theoretical maximum values. Thus, with the superscript "e" denoting exact values, we have for any mode n

$$\epsilon_\omega=(\omega^*_n/\omega^e_n)-1 \tag{43a}$$

$$\epsilon_\phi=\frac{1}{\ell\phi^e_n\,(\max)}\left[\int_0^\ell[\phi^*_n(x)-\phi^e_n\,(x)]^2\,dx\right]^{1/2} \tag{43b}$$

$$\epsilon_\psi=\frac{1}{\ell\psi^e_n\,(\max)}\left[\int_0^\ell[\psi^*_n(x)-\psi^e_n\,(x)]^2dx\right]^{1/2} \tag{43c}$$

In all cases the stress mode shapes do not employ mode acceleration. It is expected that the use of mode acceleration to obtain stresses will not change the relative accuracies of the methods.

Eigensolutions were obtained numerically for eigenvalue problem orders, p, equal to 1,2,3,4,6,10. The mode shapes for the first mode (generalized mass $=m$) derived from a first-order eigenvalue problem are shown in Fig. 4 for the various methods, along with a listing of natural frequency and the three error measures defined in Eq. (43); the exact quarter-sinusoidal shapes are also shown. The three error measures for the first three modes are shown in Fig. 5 as a function of the eigenproblem order p; the curves are only meaningful at integer values of p. As p increases, the errors in a given mode tend asymptotically to diminish in proportion to $p^{-\beta}$

$$\epsilon_\omega\propto p^{-\beta_\omega},\ \epsilon_\phi\propto p^{-\beta_\phi},\ \epsilon_\psi\propto p^{-\beta_\psi} \tag{44}$$

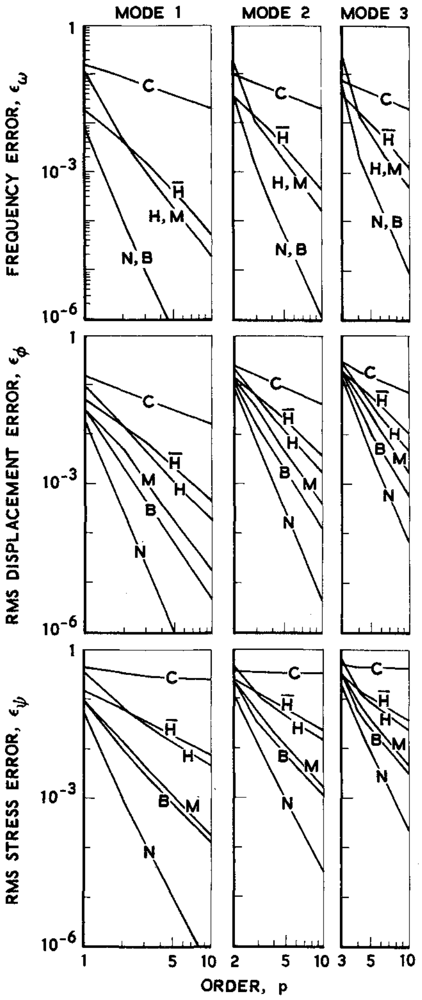


Fig. 5 Errors for lowest three cantilever modes vs order of eigenvalue problem, p.

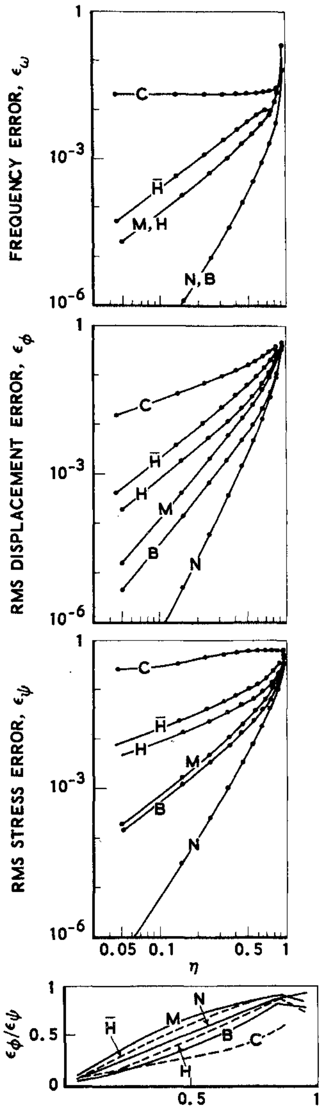


Fig. 6 Errors for $p=10$ vs frequency ratio η.

Table 1 Exponents for asymptotic convergence rates; see Eq. (44)

	β_ω	β_ϕ	β_ψ
New	5.5	5.7	4.8
Bamford	5.5	3.7	2.5
MacNeal	3.2	3.3	2.6
Hurty	3.2	2.7	1.5
Hurty variation	3.0	2.3	1.4
Classical	0.9	1.0	0.0

Table 1 contains the empirically observed values of the error exponents, β. With a few exceptions, the ranking of the various methods in terms of error magnitude at any order p is the same as the ranking according to convergence rates. *In descending order of accuracy, the ranking is new, Bamford, MacNeal, Hurty, Hurty variation, and classical*‡‡ Exceptions are that the natural frequencies for the new and MacNeal methods are identical for all p to those for the Bamford and Hurty methods, respectively. These results highlight the potential inadequacy of the common practice of basing accuracy evaluations solely on natural frequencies.

Another display of the errors would be their variation with mode number for a given eigenproblem order. Instead of the mode number, however, it is more enlightening to employ the ratio of modal frequency to the frequency of the lowest truncated mode; this ratio is designated by $\eta(0<\eta<1)$. For example, for the new, MacNeal, and Bamford methods, the lowest truncated mode for $p=10$ has a natural frequency $\omega_n = 10\pi/\tau$; the value of η for the first cantilevered mode is thus $\eta_1 = (\pi/2)/10\pi$ or 0.05; for the tenth mode, $\eta_{10} = (19\pi/2)/10\pi = 0.95$. In general, $\eta_j = (2j-1)/2p$ for the new, MacNeal, Hurty, and Bamford methods and $\eta_j = (2j-1)/2(p+1)$ for the Hurty variation and classical methods. Figure 6 contains plots of the three errors and the ratio $\epsilon_\phi/\epsilon_\psi$ vs η for the tenth-order eigenproblem ($p=10$);

‡‡Consideration of any computational advantage of matrix sparsity, such as exists for the Hurty method, is beyond the scope of this paper.

Table 2 Approximate functional relationship of errors with η for a given problem order[a]

Method	ϵ_ω	ϵ_ϕ	ϵ_ψ
New	$\eta^4/(1-\eta^2)\pm 10\%$	$\eta^5/(1-\eta^2)\pm 2\%$	$\eta^4/(1-\eta^2)\pm 1\%$
Bamford	(identical to new)	$\eta^3/(1-\eta^2)\pm 16\%$	$\eta^2/(1-\eta^2)\pm 3\%$
MacNeal	$\eta^2/(1-\eta^2)\pm 22\%$	$\eta^3/(1-\eta^2)\pm 21\%$	$\eta^2/(1-\eta^2)\pm 6\%$
Hurty	(identical to MacNeal)	$\eta^2/(1-\eta^2)\pm 8\%$	$\eta/(1-\eta^2)\pm 10\%$
Hurty variation	$\eta^2(1-\eta^3)\pm 9\%$	$\eta^2/(1-\eta^3)\pm 12\%$	$\eta/(1-\eta^3)\pm 6\%$
Classical	$\eta^0\pm 10\%$	$\eta\pm 11\%$	η^{0b}

[a]Highest mode not considered. Tolerances for $p=10$.
[b]Tendency for highest modes only.

the dots denote values for the modes. Note that the errors rank in the previously identified order, now without exception.

An empirical study of the error variations with η reveals an approximate proportionality to the functional forms shown in Table 2 if the last mode is ignored; the percentages listed are the tolerances on the relationship for $p=10$. The methods are ordered in Table 2 as they were previously ranked in accuracy. Note that the magnitude of the power of η in the lead factor correlates qualitatively with accuracy for all three errors as seen from Fig. 5; moreover, the changes in the power of η correlate roughly with the changes in the convergence rates β listed in Table 1. Interestingly, functional form for ϵ_ω and ϵ_ϕ for the new method is same as the variation of the error displacements $\tilde{q}_n^{(2)}/f_n$ with η replacing ω/ω_n; the same holds for the corresponding form of the MacNeal error and $\tilde{q}_n^{(1)}/f_n$. Note that the exponent of η for the displacement error ϵ_ϕ is always one more than that for the stress error ϵ_ψ. Except for the classical method, the factor of proportionality to the forms listed in Table 2 is roughly the same for both ϵ_ϕ and ϵ_ψ, so that (see Fig. 6)

$$\epsilon_\phi/\epsilon_\psi \approx \eta \tag{45}$$

This means that, *relative to the displacement error, the stress error is large for the lower (most accurate) modes, and roughly equal for the higher (least accurate) modes.* This is contrary to the usual expectation that the derivative of a function deteriorates in accuracy more rapidly than does the function itself.

A simple criterion for determining the number of component modes for modal synthesis with the method of this paper is to employ all modes whose natural frequency is below 1.5 times the highest frequency mode of interest (i.e., η will not exceed 2/3). Recalling Eq. (4b), the factor of 1.5 permits an error of no greater than 20% [i.e., $(2/3)^4$] in the contribution of any lightly damped mode that is not retained. Application of this criterion to the various order rod problems solved here shows that the third mode of the fourth-order problem ($\eta=0.625$) yields the largest errors: $\epsilon_\omega \approx 2.4\times10^{-3}$, $\epsilon_\phi \approx 1.9\times10^{-2}$, $\epsilon_\phi \approx 2.9\times10^{-2}$.

Some additional comments on the errors are as follows: 1) The classical method displays extremely poor accuracy, especially for stress. This is so since the stress (and thus the displacement slope) is forced to vanish at the cantilevered end because only free-boundary modes are employed (e.g., see Fig. 4).

2) As discussed in Sec. IIID, the new method yields a small nonzero value for the base displacement. The magnitude of this base displacement was irregularly scattered generally between one and two times the rms displacement error for that mode. (The mean of this factor was 1.5 with a standard deviation of 0.3.)

For the highest mode computed, the Hurty variation and classical methods yield lower frequency errors for $p \geq 3$ and $p \geq 5$, respectively, than do the new and Bamford methods (e.g., see ϵ_ω for mode 3 of $p=3$ on Fig. 5). The reason stems from the fact that the Hurty variation and classical methods employ an additional mode for a given order eigenvalue problem for this particular rod example. The Hurty variation method yields lower displacement and stress errors in the highest mode relative to the new method beyond a p of about 10.

V. Correlation of Analysis and Test

We are not concerned here with the technology for the experimental determination of the normal modes of a structure. We need only point out that free-boundary conditions are usually relatively practical to implement and, as we are about to discuss, they are compatible with a very natural and practical type of frequency-response testing to correspond with the requirements of the new analytical method. As we shall show, direct and complete correlation of analysis and test is well served. To the contrary, testing cannot practically provide a *complete* experimental definition for methods based upon the imposition of boundary constraints.

The additional testing requirement imposed by the new method for modal synthesis is the determination of the residual effects (i.e., $G_{\rho a}$, $H_{\rho a}$, $B_{\rho a}$) vs frequency at the points of applied force (boundary points and other points at which forces are applied).§§ In concept this can be accomplished as follows:

1) Determine the rigid-body modes and determine all the flexible-body modes occurring up to, say, 1.5 times the upper frequency desired for accurate modeling. This is the same factor as discussed earlier relative to the retention of component modes for an analytical modal synthesis.

2) Measure the individual elements of the dynamic-flexibility (or mechanical admittance or mobility) matrix for the points of applied force, Y_a, defined as a function of frequency by[18]

$$U_a(\omega) = Y_a(\omega) F_a(\omega) \tag{46}$$

The measurements can be made most accurately by applying one force at a time, and measuring all the motion responses U_a (acceleration, velocity, or displacement) under steady-state sinusoidal conditions.¶¶ This is done from as low a frequency as is practical up to the highest frequency of interest. The application of an individual boundary force leads to the determination of a single column of $Y_a(\omega)$, the elements of which correspond to the various individual motions.

3) The residual dynamic-flexibility matrix for the points of applied force, $Y_{\rho a}(\omega)$, is then determined by deducting the contribution of all the modes determined in Step 1 from the total dynamic-flexibility matrix, $Y_a(\omega)$, determined in Step 2. Thus

$$Y_{\rho a}(\omega) = Y_a(\omega) - \Phi_{Ra} Y_R \Phi_{Ra}^T - \Phi_{Na} Y_N \Phi_{Na}^T \tag{47}$$

where Y_R and Y_N are the modal dynamic-flexibility matrices for the rigid-body modes and for the retained flexibility-body modes, respectively

$$Y_R = -\frac{1}{\omega^2} M_R^{-1}, \quad Y_N = \Lambda_N^{-1} M_N^{-1} \tag{48}$$

where Λ_N will be diagonal with elements $-\omega^2 + i2\zeta_n \omega\omega_n + \omega_n^2$ for a sequence of classically damped modes.

4) It remains only to approximate $Y_{\rho a}(\omega)$ over the frequency range of interest by the form [corresponding to Eq. (23)]

$$Y_{\rho a}(\omega) = G_{\rho a} + \omega^2 H_{\rho a} - i\omega B_{\rho a} \tag{49}$$

and, if this second-order form provides a good fit to $Y_{\rho a}(\omega)$, we have achieved our goal. If the fit is not good, it may be due to: a) measurement errors in the modal parameters or in Y_a, b) the need to experimentally determine additional modes, or c) departures from the assumptions of linearity and classical modal damping. *The fitting with Eq. (49) thus provides a powerful check on the quality of the experimental component-mode representation.* The fitting process should take full advantage of the analyst's judgment to 1) emphasize those elements of $Y_{\rho a}$ which are most significant; and 2) emphasize results at frequencies where measurement errors tend to be relatively unimportant (e.g., near antiresonances in the summed modal response).[14,15]

The order of the frequency-response matrix Y_a equals the number of applied forces. For example, 4 boundary points

§§The experimental determination of the residual-flexibility matrix is discussed in Refs. 14 and 15.

¶¶The particular problem of measuring planar translational and rotational forces and motions at a point on the structure is discussed in Ref. 19.

with the possibility of all 6 rectilinear and rotational forces at each yields a 24th-order matrix. One may argue that the measurement of Y_a and the subsequent fitting process to obtain Y_{pa} may present so formidable a task when Y_a is of large order that the whole approach will be impractical. A reply is that we are here striving to achieve a *capability for full experimental definition* of the dynamics of the structure, and that is the reason for the measurement of Y_a. To argue that other methods do not require such extensive measurements begs the question—they do not only because their objective stops short of full experimental determination. The approach of this paper permits the full determination in terms of a straightforward experimental procedure, even though the quantity of data may make the task extensive. The analyst always has the option of applying his judgment about the particular structure in question to reduce the experimental task to what he believes is reasonable.*** It is contended that the new method gives him the freedom to exercise that option to any degree he desires.

The practicality of this experimental approach has been demonstrated with residual effects limited to the flexibility.[13,14] The experimental studies were conducted on a system consisting of a truck frame, mounts, and cab and also on a 1750-hp electric motor mounted on a foundation. It was concluded that the approach is "extremely attractive since it is practically feasible for large complex systems and parallels the substructuring design process."[14]

I. Example 3

Consider a test on the rod shown in Fig. 2. Suppose that only the first flexible-body mode ($\omega_n \tau = \pi$) is determined by modal test and that the flexibility $y_b(\omega) = u(0,\omega)/f(\omega)$ is also determined experimentally. The residual dynamic flexibility is calculated by application of Eq. (47); if one assumes no experimental error and neglects damping, the result is

$$(ES/\ell) y_{pb}(\omega) = -\frac{\cot\omega\tau}{\omega\tau} + \frac{1}{(\omega\tau)^2} - \frac{2}{\pi^2 - (\omega\tau)^2}$$

This is plotted on Fig. 7 along with the analytically predicted second-order form

$$(ES/\ell) y_{pb}(\omega) = \frac{1}{3} + \frac{(\omega\tau)^2}{45} - \frac{2}{\pi^2}\left[1 - \frac{(\omega\tau)^2}{\pi^2}\right]$$

The fit is seen to deteriorate rapidly with frequency above that of the highest mode included (at $\omega\tau/\pi = 1$). In an actual application, the frequency range over which a good match exists is based on the analysts' judgment of the permissible error.

VI. Summary and Conclusions

The new component-mode method, when employed as a building block for system analysis, satisfies in an outstanding manner the desires for computational efficiency, total independence of the component representations, and capability for complete correlation with component test data. The method employs an incomplete set of free-boundary normal modes, along with an account for "residual effects" (that is, the low-frequency flexibility, inertial, and dissipative contribution of all neglected modes). Determination of these residual effects does not require any knowledge whatsoever of the neglected modes, but merely requires the solution of a special statics problem along with the characteristics of the retained modes. An equivalency is shown between the benefit of including the residual effects and the benefit of the application of the mode-acceleration method. A conservative criterion for the number of component modes needed is to employ all modes whose natural frequency is below 1.5 times the highest frequency of interest.

***For example, he might judge that only the diagonal elements and a relatively few off-diagonal elements are significant.

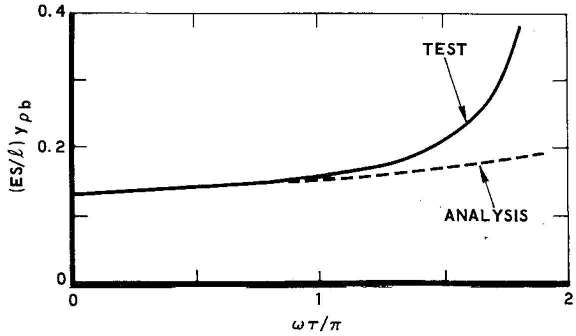


Fig. 7 Residual dynamic flexibility y_{pb} vs frequency.

The method is applicable to both distributed and lumped-parameter representations of components and to any degree of redundancy among the component boundary attachment points. The representation for each component can, if desired, be put into a standard stiffness-matrix form when the number of independent degrees of freedom in the statics problem is greater than the number of boundary degrees of freedom [see Eq. (30)].

Numerical results of modal analysis for a cantilevered, uniform, undamped rod were obtained for several methods utilizing free-boundary modes (new, MacNeal, classical, Bamford) as well as with fixed-boundary modes (Hurty). This rod problem provides an equally severe test for the two types of boundary conditions, and it is believed to be a critical test of capability for modal syntheses. With minor exceptions, the ranking of the methods in descending order of accuracy is new, Bamford, MacNeal, Hurty, and classical. Several interesting characteristics of the errors are described.

In general, the number of flexible-body normal modes employed for a given order system eigenproblem is greatest for a method like that of Benfield and Hruda;[2,9] intermediate in number for the Hurty method; and smallest for the new, MacNeal, or Bamford methods. The difference in the first instance increases with the number of interconnection coordinates and in the second instance with the number of component rigid-body modes. As stated in Ref. 3, when there are relatively large numbers of interconnection coordinates, the Hurty method suffers in accuracy relative to the method of Benfield and Hruda. Likewise, the new method suffers relative to the Hurty method when there are a relatively large number of component rigid-body modes.

A compelling feature of the new method is the possibility of completely basing the representation on test data. Consequently, experimental to analytical correlation of all individual parameters is possible for a component representation. The testing requires the determination of free-boundary normal modes and the dynamic-flexibility matrix for the boundary points. The subsequent determination of the boundary residual effects requires curve fitting using second-order polynomials in frequency. The departures of the data from such a fitting provides a means for evaluation of the overall quality of the test-derived representation. Although successful results have been obtained in limited experimental studies, more must be done to fully demonstrate the practicality of the experimental method.

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